



An improved harmonic map ansatz

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Abstract

The rational map ansatz of Houghton et al. [Nucl. Phys. B 510 (1998) 507] is generalised by allowing the profile function, usually a function of r , to depend also on z and \bar{z} . It is shown that, within this ansatz, the energies of the lowest $B = 2, 3, 4$ field configurations of the $SU(2)$ Skyrme model are closer to the corresponding values of the true solutions of the model than those obtained within the original rational map ansatz. In particular, we present plots of the profile functions which do exhibit their dependence on z and \bar{z} .

The obvious generalisation of the ansatz to higher $SU(N)$ models involving the introduction of more projectors is briefly mentioned.

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1. Introduction

A few years ago Houghton et al. [1] presented an ansatz, the so-called rational map ansatz, in order to approximate multiskyrmion solutions of the $SU(2)$ Skyrme model by field configurations in which the angular dependence was determined by a rational map and the radial dependence was determined by a numerical solution of a nonlinear ordinary differential equation. This last equation involved the so-called profile function (i.e., the generalisation of the one skyrmion

profile function) and its shape had to be determined numerically.

This ansatz has been a great step forward since it gave very good approximations to the solutions of the full equations which up to then could only be determined numerically (and these simulations involved hours, days or weeks of CPU time). In fact, these approximations were so good that the values of the energies were only, at most, a few % up on the true value and it was practically impossible to distinguish the energy density plots obtained with the use of the ansatz from the exact ones obtained numerically.

At the same time the ansatz clarified the situation for higher baryon number states; the energy density had maxima at several points lying on a shell

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whose radius grew with the baryon number. It has also lead to some generalisations—first, the generalisation to the $SU(N)$ Skyrme models lead to the harmonic map ansatz [2], in which the original rational map of Houghton et al. [1] was replaced by the projector of the more general harmonic map of the CP^{N-1} model and then to the realisation that one could use more of such projectors, which represent harmonic maps, to obtain further solutions (radial cases) or field configurations of the $SU(N)$ models. Since then, these ideas have been used to other models (i.e., monopoles) or similar ones coupled to gravity.

All these studies have relied on one fundamental assumption: the separation of the angular degrees of freedom given in terms of a projector or a series of projectors and the radial dependence built in through the profile function or functions (for more projectors).

However, it is easy to check that the profile function does not have to depend only on r ; it could also depend on the angular variables. As long as the projectors represent *harmonic maps* the ansatz keeps some of its useful features, like partial factorization. Employing more general profile functions on the other hand, provides a better approximation to the solutions of the model. This observation constitutes the essence of our *improved harmonic map ansatz*; the profile functions in addition to their dependence on r do depend, also, on z and \bar{z} which allows for a more general angular dependence of the fields and density functions.

In this Letter we present our ansatz and reinvestigate the lowest baryon number field configurations of the $SU(2)$ model. In particular we look at $B = 2, 3$ and 4. In the subsequent papers we will look at larger values of B ; the $SU(N)$ models for $N > 2$; and the multiprojector case. We will also look at the gravitating cases involving skyrmions and monopoles.

2. The harmonic map ansatz and its improved version

The $SU(N)$ Skyrme action to be considered is given by

$$S = \int \left[\frac{\kappa^2}{4} \text{tr}(K_\mu K^\mu) + \frac{1}{32e^2} \text{tr}([K_\mu, K_\nu][K^\mu, K^\nu]) \right] d^4x, \quad (1)$$

where $K_\mu = \partial_\mu U U^{-1}$ for $\mu = 0, 1, 2, 3$; U is the $SU(N)$ chiral field and κ, e are coupling constants. Although, we present the discussion for the general $SU(N)$ model in this Letter we will give the details of our studies for the $SU(2)$ case.

It is well known that, in order for the finite-energy configurations to exist, the Skyrme field has to go to a constant matrix at spatial infinity: $U \rightarrow I$ as $|x^\mu| \rightarrow \infty$. This effectively compactifies the three-dimensional Euclidean space into S^3 and hence implies that the field configurations of the Skyrme model can be considered as maps from S^3 into $SU(N)$.

This compactification leads to the existence of a conserved topological current yielding the topological charge to be identified with the baryon number B defined as

$$B = \int B^0 d^3x, \quad (2)$$

where

$$B^\mu = -\frac{1}{24\pi^2} \varepsilon^{\mu\nu\alpha\beta} \text{tr}(K_\nu K_\alpha K_\beta) \quad (3)$$

and $\varepsilon^{\mu\nu\alpha\beta}$ is the (constant) fully antisymmetric tensor.

The starting point of the further discussion is the introduction of the coordinates r, z, \bar{z} on \mathbb{R}^3 . In terms of the usual spherical coordinates r, θ, ϕ the Riemann sphere variable z is defined by: $z = e^{i\phi} \tan(\theta/2)$. Then, the Skyrme action becomes [1]

$$S = \int dr dt dz d\bar{z} \times \text{tr} \left(\frac{\kappa^2 r^2}{2(1+|z|^2)^2} K_r^2 + \frac{\kappa^2}{2} |K_z|^2 + \frac{1}{8e^2} |[K_r, K_z]|^2 - \frac{(1+|z|^2)^2}{32e^2 r^2} [K_z, K_{\bar{z}}]^2 \right) \quad (4)$$

while the baryon number takes the form

$$B = -\frac{1}{8\pi^2} \int \text{tr}(K_r [K_z, K_{\bar{z}}]) dr dz d\bar{z}. \quad (5)$$

Since we are interested in the static field configurations in what follows we assume no t dependence.

Next we consider the harmonic map ansatz which involves assuming that the Skyrme field is of the form [2]

$$U = e^{2ih(P-I/N)} = e^{-2ih/N} [I + (e^{2ih} - 1)P], \quad (6)$$

where P is a $N \times N$ hermitian projector which depends only on the angular variables (z, \bar{z}) and h is the profile function which depends, at least, on r . Note that, the matrix P can be thought of as describing a mapping from S^2 into CP^{N-1} defined as

$$P(V) = \frac{V \otimes V^\dagger}{|V|^2}, \quad (7)$$

where V is a N -component complex vector (depending on z and \bar{z}). For $N = 2$ and $V = (1, f(z))$ where $f(z)$ is a rational function we recover the rational map ansatz of Houghton et al. [1].

Following [3], we define a new operator P_+ by its action on any vector $v \in \mathbb{C}^N$ as

$$P_+ v = \partial_z v - \frac{v(v^\dagger \partial_z v)}{|v|^2}. \quad (8)$$

Note that, when $V = V(z)$ (only) then

$$P_z = \frac{P_+ V \otimes V^\dagger}{|V|^2} \quad (9)$$

and so

$$\begin{aligned} P P_z &= 0, & P_z P &= P_z, \\ P_{\bar{z}} P &= 0, & P P_{\bar{z}} &= P_{\bar{z}}, \end{aligned} \quad (10)$$

where the subscripts denote partial derivatives.

For (6) to be well-defined at the origin, the profile function h , as a function of r , has to satisfy $h(0) = \pi$ while the boundary value $U \rightarrow I$ at $r = \infty$ requires that $h(\infty) = 0$. As shown in [1,2], an attractive feature of (6) is that it leads to a simple expression for the energy density which can be successively minimized with respect to the parameters of the projector P and then with respect to the shape of the profile function h . This procedure for $h = h(r)$ gives good approximations to multiskyrmion field configurations [1,2].

In what follows, this ansatz is “improved” by allowing the profile function h to depend on z and \bar{z} in addition to its r dependence. The new ansatz is consistent with the partial factorisation of the field in the sense that it still reduces the problem to having to solve one equation for one function—namely h . Had we taken an ansatz in which, say, the parameters of the projector P depended on r , such a simplification would not

have taken place. Thus, this modification is “nontrivial” and that is the reason for calling it an *improved harmonic map ansatz*. Note that, h has to be real implying that $h = h(r, |z|^2, \frac{z+\bar{z}}{|z|})$; while, at the origin, we require that $h(0, |z|^2, \frac{z+\bar{z}}{|z|}) = \pi$.

The action (4), due to (6) for $h = h(r, z, \bar{z})$ and using the aforementioned properties of the harmonic maps, becomes

$$\begin{aligned} S = \int dt dr dz d\bar{z} & \times \left(-\kappa^2 A_N r^2 h_r^2 - \kappa^2 B_N |h_z|^2 \right. \\ & - \left[\mathcal{N}_1 \left(\kappa^2 + \frac{h_r^2}{e^2} \right) + \frac{\mathcal{N}_2 |h_z|^2}{e^2 r^2} \right] \sin^2 h \\ & \left. - \frac{\mathcal{I} \sin^4 h}{e^2 r^2} \right), \end{aligned} \quad (11)$$

where

$$\begin{aligned} A_N &= 2i \frac{N-1}{N} \frac{1}{(1+|z|^2)^2}, & B_N &= 2i \frac{N-1}{N}, \\ \mathcal{N}_1 &= i 2 \frac{|P_+ V|^2}{|V|^2}, & \mathcal{N}_2 &= i \frac{|P_+ V|^2}{|V|^2} (1+|z|^2)^2, \\ \mathcal{I} &= i \frac{|P_+ V|^4}{|V|^4} (1+|z|^2)^2 \end{aligned} \quad (12)$$

while the baryon number (5) coincides with the expression for the topological charge of the CP^{N-1} sigma model (up to an overall profile dependent factor) since

$$\begin{aligned} B &= \frac{i}{\pi^2} \int \text{tr}(P[P_z, P_{\bar{z}}]) dz d\bar{z} \int_0^\infty \sin^2 h h_r dr \\ &= \frac{i}{2\pi} \int \frac{|P_+ V|^2}{|V|^2} dz d\bar{z}. \end{aligned} \quad (13)$$

It is easy to see that $\int_0^\infty \sin^2 h h_r dr = h(r=0)/2 \equiv \pi/2$.

Next, for convenience of our numerical simulations, we use spherical coordinates (r, θ, ϕ) and introduce the dimensionless coordinate $x = \epsilon \kappa r$. Variation

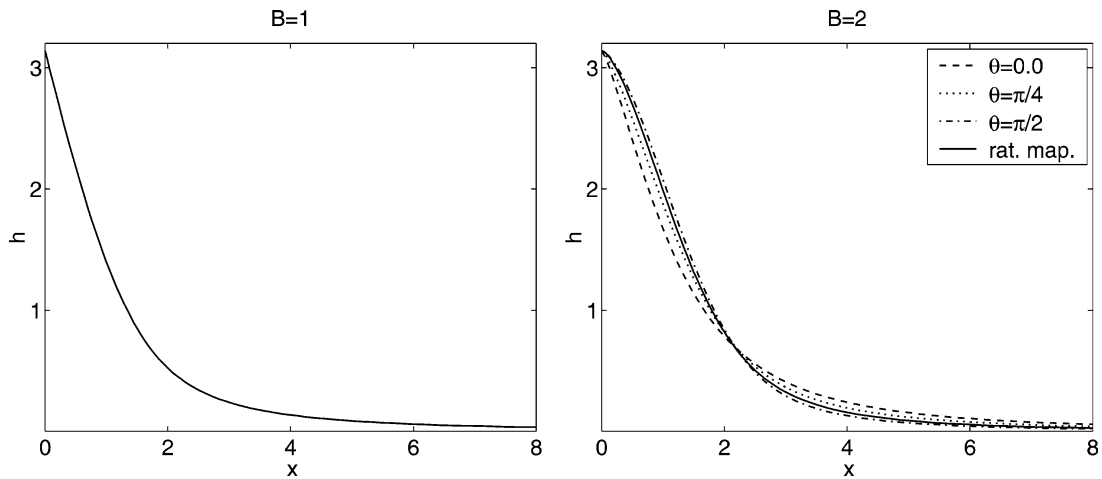


Fig. 1. The one skyrmion profile function showing no θ dependence (left) and the two skyrmion profile function obtained from the improved harmonic and rational map ansatz (right).

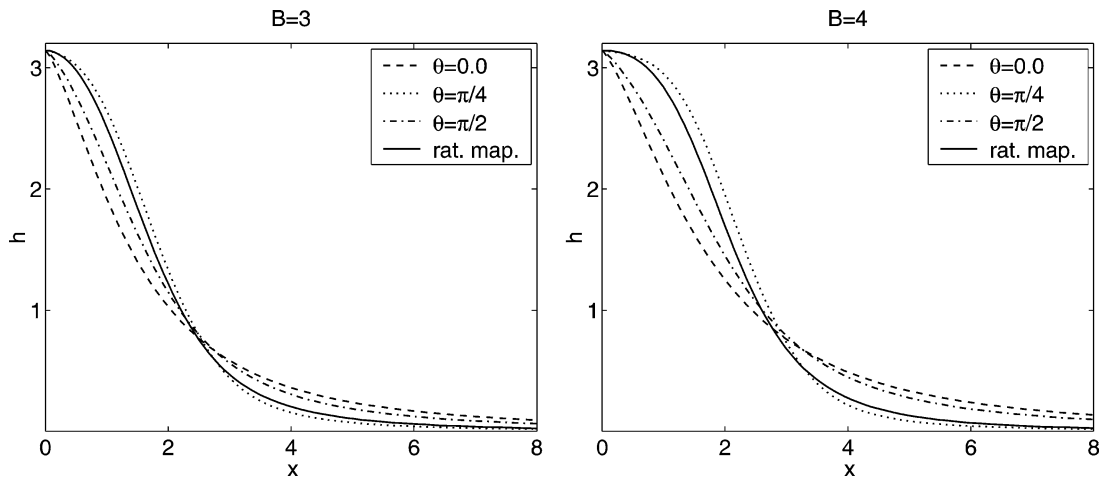


Fig. 2. The skyrmion profile function $h(r, \theta)$ with axial symmetry obtained from the improved harmonic and rational map ansatz for $B = 3$ (left) and $B = 4$ (right).

Table 1

Energies per skyrmion (i.e., E/B) of multiskyrmion configurations obtained by the rational and the improved harmonic map ansatz (with and without ϕ dependence) for $B = 1, \dots, 4$ in comparison with the energies of the “exact” solutions

B	Axial symmetry			Platonic symmetry			
	Rat. map	“Imp.” harm. map	Exact [6]	Rat. map	“Imp.” harm. map (no ϕ)	(With ϕ)	Exact [4]
1	1.232	1.232	1.232	—	—	—	—
2	1.208	1.191	1.181	—	—	—	—
3	1.256	1.214	1.194	1.184	1.183	1.168	1.143
4	1.322	1.243	1.216	1.137	1.133	1.130	1.116

of (11) with respect to h gives the equation of motion

$$\begin{aligned} & \partial_x \left(\left[\frac{2(N-1)}{N} + \frac{2\sin^2 h}{x^2} G \right] h_x x^2 \sin \theta \right) \\ & + \partial_\theta \left(\left[\frac{2(N-1)}{N} + \frac{\sin^2 h}{x^2} G \right] h_\theta \sin \theta \right) \\ & + \partial_\phi \left(\left[\frac{2(N-1)}{N} + \frac{\sin^2 h}{x^2} G \right] \frac{h_\phi}{\sin \theta} \right) \\ & - \left(1 + \frac{h_r^2}{2} + \frac{h_\theta^2}{2x^2} + \frac{h_\phi^2}{2x^2 \sin^2 \theta} + \frac{\sin^2 h}{x^2} G \right) \\ & \times \sin(2h) G \sin \theta = 0, \end{aligned} \quad (14)$$

where $G = \frac{|P_+ V|^2}{|V|^2} (1 + |z|^2)^2$ is a function of θ and ϕ only.

3. $SU(2)$ $B = 1, \dots, 4$ baryons

First, to test our approach, we have calculated the profile function h for one skyrmion (i.e., for $B = 1$) and found no θ or ϕ dependence (as expected), as Fig. 1 indicates. The total energy is 1.232.

Axially symmetric skyrmions with baryon number $B > 1$ can be obtained from vectors of the form [1]

$$V = (z^B, 1)^t. \quad (15)$$

In this case, the action (11) depends explicitly on θ but not on ϕ . Consequently, we find θ dependence of the corresponding profile functions h , as shown in Figs. 1 and 2. The energies are given in Table 1 and compared with those obtained from the rational map ansatz. For $B = 2$ the energy is 2.382 which is closer to the value 2.342 of the “exact” solution (obtained numerically by solving the full equations [4]) and lower than the value 2.416 obtained by the rational map ansatz [1]. (In this case, h is ϕ independent (as expected).) For $B > 2$ the solutions (of this class) correspond to saddle points of the energy. Obviously, the improved harmonic map ansatz yields lower energies than the rational map ansatz, as can be seen in Table 1.

Next, we consider skyrmions with platonic symmetry. For $B = 3$ and $B = 4$ we let the vector V to be

$$\begin{aligned} V &= (\sqrt{3}iz^2 - 1, z(z^2 - i\sqrt{3}))^t, \\ V &= (z^4 + 2\sqrt{3}iz^2 + 1, z^4 - 2\sqrt{3}iz^2 + 1)^t. \end{aligned} \quad (16)$$

All these expressions come from [1]—who obtained their form from symmetry arguments. In [5] it was established that the same expressions minimize the projector part of the action. This time, the corresponding action (11) depends explicitly on θ and ϕ .

First we consider the question whether we can obtain an improvement of the energy by considering h to depend (only) on θ , in addition to r . We have performed the integration of the action over the azimuthal angle ϕ which has given us an effective action, which still contains an explicit θ dependence. The variation of this effective action with respect to $h(r, \theta)$ led to a partial differential equation for it. We have found that the θ dependence of h is very small for $B = 3$ while for $B = 4$ it is more pronounced, but still small compared to the axially symmetric solution. As shown in Table 1, the improvement of the energy is rather small, compared to the improvement of the energy for the axisymmetric solutions.

This result suggests that a considerable improvement of the energy can only be achieved if we allow h to depend also on ϕ , in addition to r and θ . The variational equation then yields a second order partial differential equation involving the independent variables x, θ, ϕ . We solved this equation for $B = 3$ and $B = 4$ and found indeed a further improvement of the energies (see Table 1). For $B = 3$, respectively, $B = 4$ the deviation from the exact value is only $\approx 2\%$, respectively, $\approx 1\%$. To demonstrate the angular dependence of the profile function we plot in Figs. 3 and 4

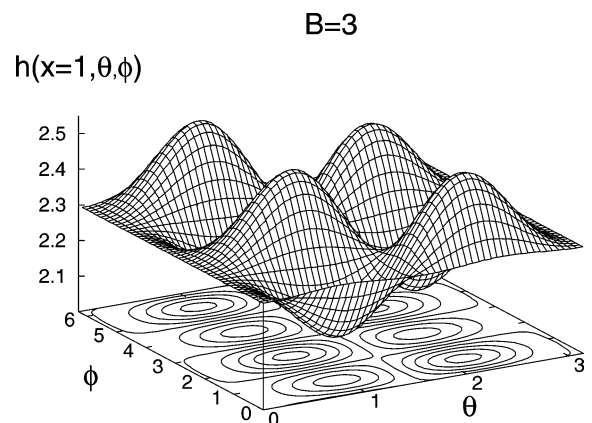


Fig. 3. The $B = 3$ skyrmion profile function $h(r = 1, \theta, \phi)$ with tetrahedral symmetry obtained from the improved harmonic map ansatz.

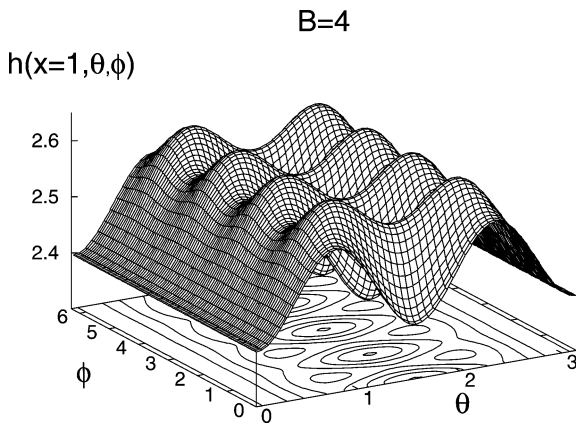


Fig. 4. Same as Fig. 3 for $B = 4$ with octahedral symmetry.

$h(x, \theta, \phi)$ for fixed $x = e\kappa r = 1$ for $B = 3$ and $B = 4$, respectively.

4. Conclusions

In this Letter we have pointed out that the rational map ansatz of Houghton et al. [1] can be further improved. The improvement involves the allowance of the profile function h to depend on z and \bar{z} in addition to r . Of course, the rational map ansatz is already a very good approximation; hence our improvement is only modest but, as we have shown (in the

cases studied) the improvement is nonnegligible. To get our values we first have assumed only θ dependence of h ; however in order to get better values for the skyrmions with platonic symmetries we went further and allowed, also, its ϕ dependence. It is interesting to note that an improvement of the energies of similar magnitude can also be obtained with nonholomorphic rational maps [7], while restricting to profile functions that only depend on r . Our modification of the ansatz is not restricted to one projector. It is easy to see that if we want to obtain low energy configurations of the $SU(N)$ model we could use more projectors [2] (up to $N - 1$ for the $SU(N)$ case) with the corresponding profile functions dependent on r , θ and ϕ . We are also looking at such further applications of our ideas: to systems involving $SU(N)$ Skyrme models for $N > 2$ and to systems involving gravitational fields.

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